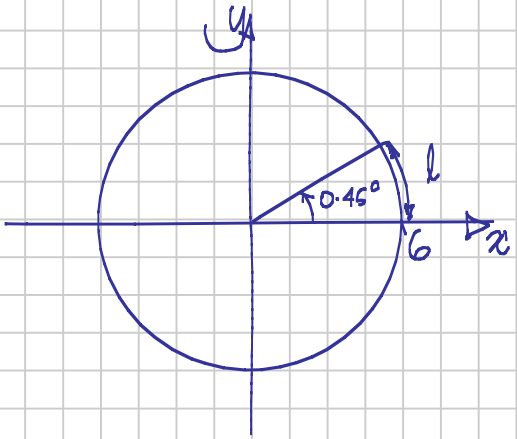
C2 exercise 6B (arc length)

Note Title 07/03/2013

1 An arc AB of a circle, centre O and radius r cm, subtends an angle θ radians at O. The length of AB is l cm.

a Find l when $r = 6$, $\theta = 0.45$

(i)



$l = r\theta$

$l = 6 \times 0.45$

$l = 2.7$

(ii) $r = 4.5$, $\theta = 0.45$

$l = r\theta$

$l = 4.5 \times 0.45$

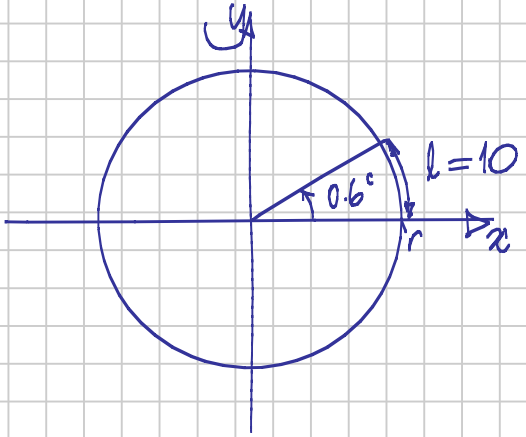
$l = 2.025$

(iii) $r = 20$, $\theta = \frac{3\pi}{8}$

$l = r\theta = \cancel{20}^{5} \times \frac{3\pi}{\cancel{8}_{2}} = \frac{15\pi}{2}$

16 Find r when

(i) $l = 10$, $\theta = 0.6$



$l = r\theta \Rightarrow r = \frac{l}{\theta}$

$r = \frac{10}{0.6} = 16\frac{2}{3}$ ✓

$r \simeq 16.7\text{ cm}$

(ii) $l = 1.26$, $\theta = 0.7$

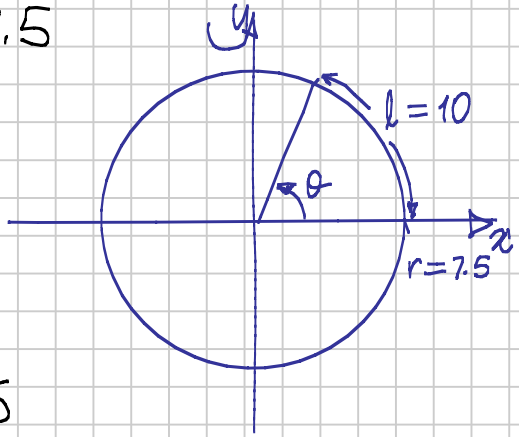
$r = \frac{l}{\theta} = \frac{1.26}{0.7} = 1.8$ ✓

(iii) $l = 1.5\pi$, $\theta = \frac{5}{12}\pi$

$r = \frac{l}{\theta} = \frac{1.5\pi}{\frac{5}{12}\pi} = \frac{3}{2} \times \frac{12}{5} = \frac{18}{5} = 3.6$ ✓

1c Find θ when

(i) $l = 10$ and $r = 7.5$



$l = r\theta$

$\Rightarrow \theta = \frac{l}{r}$

$\theta = \frac{10}{7.5} = \frac{4}{3}$

or $\theta = 1\frac{1}{3}$ ✓

(ii) $l = 4.5$, $r = 5.625$

$\theta = \frac{l}{r} = \frac{4.5}{5.625}$

$\theta = \frac{9}{2} \div \frac{45}{8} = \frac{\cancel{9}^{1}}{\cancel{2}_{1}} \times \frac{\cancel{8}^{4}}{\cancel{45}_{5}} = \frac{4}{5}$

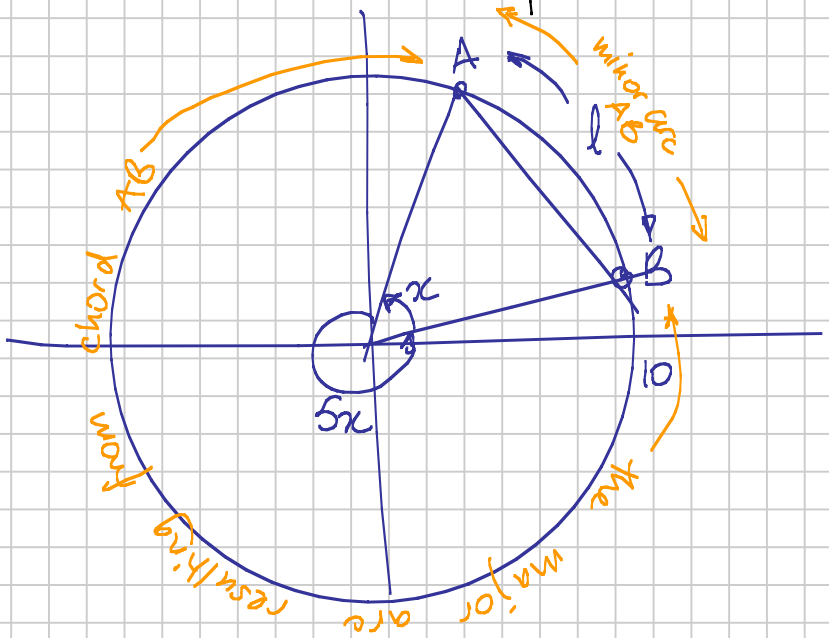
$\theta = 0.8$ ✓

(iii) $l = \sqrt{12}$, $r = \sqrt{3}$

$\theta = \frac{l}{r} = \frac{\sqrt{12}}{\sqrt{3}}$

$\theta = \frac{2\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}} = 2$

2 A minor arc AB of a circle, centre O and radius 10cm, subtends an angle x at O. The major arc AB subtends an angle $5x$ at O. Find in terms of π, the length of the minor arc AB.



A & B are on a circle radius 10 centre O.

$x + 5x = 2\pi$ (the major + minor arcs make the whole circumference)

$\Rightarrow \quad 6x = 2\pi$

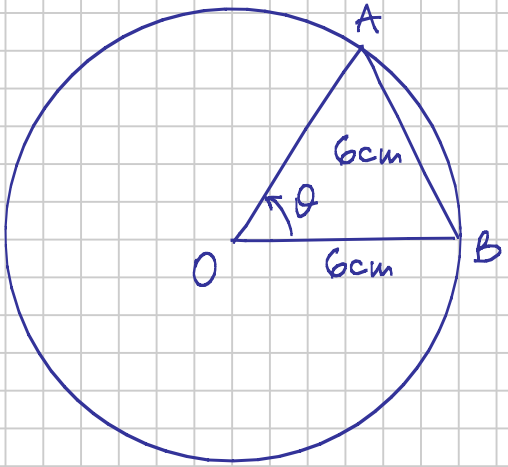
$\Rightarrow \quad x = \frac{\pi}{3}$

minor arc $l = r\theta$

$l = 10 \times \frac{\pi}{3}$

$l = \frac{10\pi}{3}$.

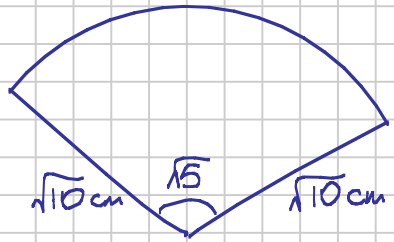
3 An arc AB of a circle, centre O and radius 6cm, has length l cm. Given that the chord AB has length 6cm, find the value of l in terms of π.



$\triangle OAB$ is equilateral $\Rightarrow \theta = 60° = \frac{\pi}{3}^c$

$l = r\theta = 6 \times \frac{\pi}{3} = 2\pi$ ✓

4 The sector of a circle of radius $\sqrt{10}$ cm contains an angle of $\sqrt{5}$ radians. Find the length of the arc giving your answer in the form $p\sqrt{q}$ cm, $(p, q \in \mathbb{N})$

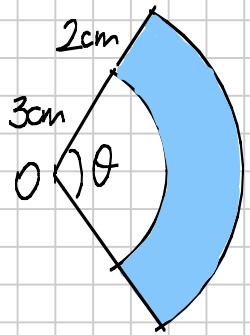


$l = r\theta$

$l = \sqrt{10} \times \sqrt{5}$

$l = \sqrt{50} = 5\sqrt{2}$. ✓

5 In the diagram find the perimeter of the shaded region when $\theta = 0.8$ radians



outer arc length $l_1 = r\theta = 5 \times 0.8 = 4$

inner arc length $l_2 = r\theta = 3 \times 0.8 = \frac{12}{6} = 2.4$

total perimeter $= l_1 + l_2 + 2 \times 2 = 4 + 2.4 + 2 + 2$

$= 10.4\text{cm}$

b In the same diagram find θ when the perimeter of the shaded shape is 14cm.

$$P = 2 \times 2 + 5\theta + 3\theta$$

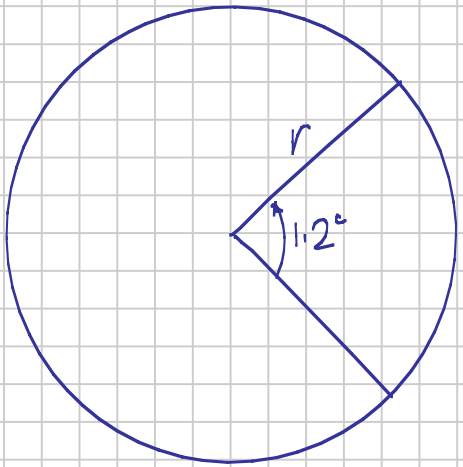
$$\Rightarrow \quad 14 = 4 + 8\theta$$

$$\Rightarrow \quad 10 = 8\theta$$

$$\Rightarrow \quad \theta = \frac{10}{8} = 1.25^c$$

$$\theta = 1\tfrac{1}{4}^c \quad \checkmark$$

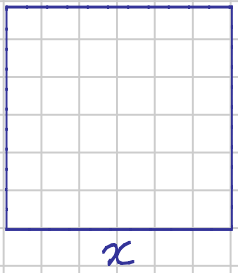
6 A circle of radius r cm contains an angle of 1.2 radians. Given that the sector has the same perimeter as a square of area 36 cm^2, find the value of r.



$P = 2r + r\theta$

$= 2r + 1.2r$

$= 3.2r$



$A = 36 \text{ cm}^2$

$x^2 = 36$

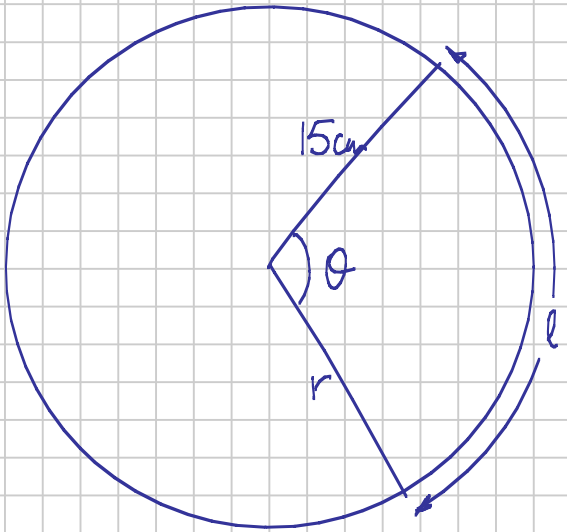
$x = 6$

$P = 4 \times 6 = 24 \text{ cm}$

$\Rightarrow \quad 24 = 3.2r$

$r = \frac{24}{3.2} = 7\frac{1}{2} \text{ cm}$

7 A sector of a circle of radius 15cm contains an angle of θ radians. Given that the perimeter of the sector is 42cm, find the value of θ.



$$P = 2r + l$$

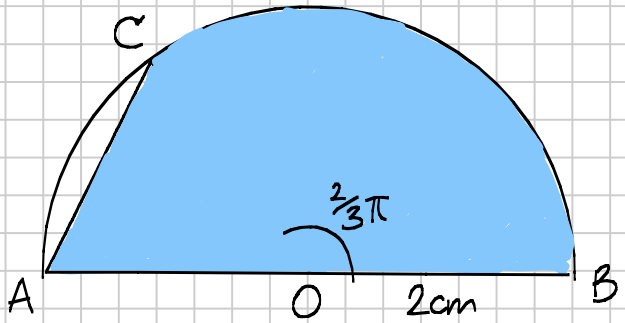
$$P = 2 \times 15 + 15\theta$$

$$42 = 30 + 15\theta$$

$$\Rightarrow \theta = \frac{12}{15}$$

$$\Rightarrow \theta = \frac{4}{5} = 0.8^c$$

8 In the diagram AB is the diameter of a circle, centre O and radius 2cm. The point C is on the circumference such that $\angle COB = \frac{2}{3}\pi$ radians.



a) State the value, in radians, of $\angle COA$

$$C\hat{O}A + C\hat{O}B = \pi \Rightarrow C\hat{O}A = \frac{\pi}{3}.$$

The shaded region enclosed by the chord AC, arc CB and AB is the template for a brooch.

b) Find the exact value of the perimeter of the brooch.

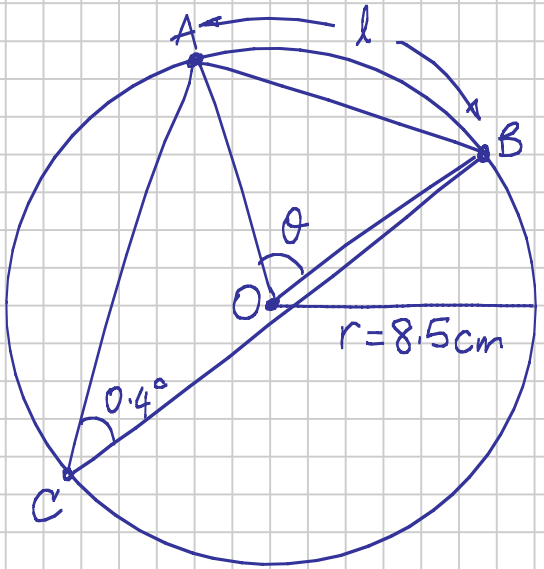
give the answer in fraction-form in terms of π

Since $A\hat{O}C = \frac{\pi}{3}$ then $\triangle AOC$ is equilateral and $AC = 2cm$

Arc $CB = r\theta = 2 \times \frac{2}{3}\pi = \frac{4}{3}\pi$

Total perimeter $= 2 + 2 + \frac{4}{3}\pi + 2 = 6 + \frac{4}{3}\pi$ ✓

9 The points A and B lie on the circumference of a circle with centre O and radius 8.5cm. The point C lies on the major arc AB. Given that $\angle ACB = 0.4$ radians, calculate the length of the minor arc AB.



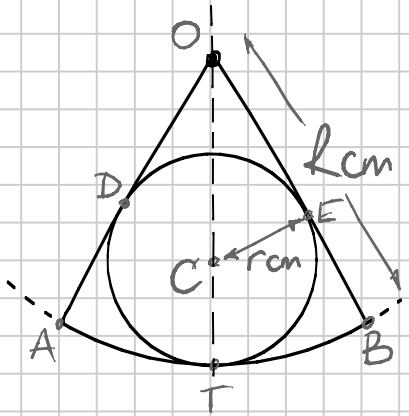
$\angle ACB = 0.4^c \Rightarrow \angle A\hat{O}B = 0.8^c$

since the angle subtended by a chord at the centre of a circle is twice the angle subtended at the circumference

(The "Star-Trek theorem")

$$l = r\theta = 8.5 \times 0.8$$
$$= 6.8\text{cm}$$

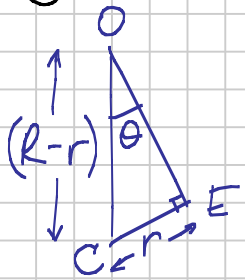
10 In the diagram OAB is a sector of a circle, centre O and radius R cm. The angle $\angle AOB = 2\theta$ radians. A second circle, centre C and radius r cm touches the arc AB at T and touches OA and OB at points D and E respectively.



a) Write down, in terms of R and r the length of OC.

The small circle has radius r
so $CT = r$.
The big circle has radius R
so $OT = R$
Hence $OC = OT - CT = R - r$.

b) Using $\triangle OCE$, show that $R\sin\theta = r(1 + \sin\theta)$



Since $A\hat{O}B = 2\theta$, $C\hat{O}E = \theta$

Now $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ $\Rightarrow \sin\theta = \frac{r}{R-r}$

$\Rightarrow (R-r)\sin\theta = r$

$\Rightarrow R\sin\theta - r\sin\theta = r$

10b) (ctd)

$$\Rightarrow \quad R\sin\theta = r + r\sin\theta$$

$$\Rightarrow \quad R\sin\theta = r(1+\sin\theta)$$

10c) Given that $\sin\theta = \frac{3}{4}$ and that the perimeter of the sector OAB is 21 cm, find r.

$$R\sin\theta = r(1+\sin\theta)$$

$$\Rightarrow \quad \frac{3}{4}R = r\left(1+\frac{3}{4}\right)$$

$$\Rightarrow \quad \frac{3}{4}R = \frac{7}{4}r$$

$$\Rightarrow \quad r = \frac{3}{7}R$$

Perimeter of sector OAB $= 2R + R(2\theta)$

$$\Rightarrow \quad 21 = 2R(1+\theta)$$

$$\Rightarrow \quad R = \frac{21}{2(1+\theta)}$$

So $r = \frac{3}{7} \times \frac{21}{2(1+\theta)}$

$$r = \frac{9}{2\left(1+\sin^{-1}\left(\frac{3}{4}\right)\right)} = 2.434983138$$

$r \simeq 2.43$ cm to 3 s.f.